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Non-selective Measurement in a Tripartite Quantum System

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Abstract

In a tripartite quantum system, research has shown how the dynamics can create correlations and entanglement especially for indirect correlation between the particles. Although, there is no direct correlation between the particles in some cases, however, after evolution or transformation they become entangled. In this paper, we demonstrate the use of one type of von Neumann measurement called the non-selective measurement. The objective of this paper is to investigate the effects of non-selective measurement on a chain of three oscillators A, B, C with nearest neighbour coupling when we performed the photon number measurement on A. We are interested to investigate the effects of quantum measurement on the correlation and entanglement, that is, whether it will increase or decrease and also the effects on the whole tripartite quantum system. It was discovered that, after the measurement on A, the correlation of oscillators A, B and A, C collapse to factorizable states and do not effect the entanglement of oscillator B,C.

Keywords: quantum measurement; von Neumann measurement; non-selective measurement; correlation; quantum entanglement

1. Introduction

Quantum correlation or entanglement is the most spectacular and counter intuitive manifestation of quantum mechanics [1]. It describes the way that particles such as photons, electrons or qubits correlate with each other regardless of how far apart they are. The dynamics to create correlations and entanglement have been studied by Refs. [2-4]. In Ref. [2], the authors consider three oscillators *A*, *B* and *C* with oscillators *B* and *C* interacting with each other, while the oscillator *A* does not interact with any of them. They show that after the evolution the

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oscillators A, C become entangled. Meanwhile in Ref. [3], a chain of three oscillators A, B, C with nearest neighbour coupling is considered. The oscillators A, C are not coupled directly. This example gave two counter intuitive results after the time evolution. The two counter intuitive results are quantum mutual information (I_{ij}) between oscillators A, B and A, C i.e. $I_{AC} > I_{AB}$ and oscillators A, C is entangled but inconclusive for oscillators A, B although there is no direct coupling between oscillators A, C. The correlations between A, C can be stronger than the correlations between A, B and in some cases A, C is entangled but we cannot conclude that A, B are also entangled. Interestingly, in Ref. [5] another result base on monogamy of entanglement, which says that when two partners A and B are more entangled then either one of them has to be less entangled with the third party. It present qualitatively the converse monogamy of entanglement where when A and B are weakly entangled, then either one of them is generally strongly entangled with the third party.

In quantum physics, the measurement process or the process to extract quantum information from a quantum system, will “collapse” the state of the system [6, 7]. This is one of the peculiarities in quantum mechanics that is interesting to study.

In the present paper, we are interested to investigate a chain of three oscillators A, B, C in Ref. [3] which give two counter intuitive results. Our objectives are to study how the collapse state due to the measurement on A effects the whole state of the systems and to look whether we still can get a counter-intuitive result after the measurement.

A few study have been done to describe the relation between measurement and entropy, measurement and correlation and also entanglement [2, 8, 9]. Further discussion about this relation will be discussed in Sec. 3.

In Sec. 2 we described the von Neumann measurement in which we use non-selective measurement. In Sec. 3, further discussion about the relation between measurement and entropy, measurement and correlation and measurement and entanglement is given. Sec. 4, explains the example that we considered, the Hamiltonian and all the formulae that we used. Meanwhile, in Sec. 5, we provide numerical results of the measurement. We discuss our results in Sec. 6.

2. Von Neumann Measurement

In quantum mechanics, the act of measurement generally changes the state of the system [7]. Measurement on a quantum system is given by a postulate [7, 8]. If the state of the quantum system before the measurement is $|\psi\rangle$ then the state of the system after the measurement is

$$|\psi_m{}'\rangle = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^{+}M_m|\psi\rangle}} \tag{1}$$

where $\langle\psi|M_m^{+}M_m|\psi\rangle$ is the probability of result m occurring.

Two types of von Neumann measurement, selective and non-selective measurements [10, 11]. For selective measurement, the results are communicated instantaneously with classical methods to the others oscillators. Meanwhile, for non-selective measurement, it is known that a measurement has been made, but the results or outcomes are not known. In this paper we only focus on non-selective measurement.

The state immediately known after the non-selective measurement is given by

$$\tilde{\rho}_{ABC}(t) = \sum_{i=0}^{d-1} \Pi_i \rho_{ABC}(t) \Pi_i \tag{2}$$

where $\rho_{ABC}(t)$, the state just before measurements and Π_i is the projection operators.

3. Relations of Measurement

The relation of measurement with entropy, correlation and entanglement have been found from the theorem and previous study. One of the theorem given in Ref. [8] is the effect of entropy of a quantum system when a non-selective measurement is performed to the subpart of the quantum system. This theorem is as follows:

Theorem 1. (*von Neumann measurement increase entropy*)
Suppose Π_N *is a complete set of orthogonal projectors and* ρ *is a density operator. Then the entropy of the state* $\tilde{\rho} = \Sigma_N \Pi_N \rho \Pi_N$ *of the system after the measurement* $\left(S\left(\tilde{\rho}\right)\right)$ *is at least as great as the original entropy* $\left(S\left(\rho\right)\right)$

$$S\left(\tilde{\rho}\right) \geq S\left(\rho\right) \tag{3}$$

with equality if and only if $\rho = \tilde{\rho}$.

Base on Ref. [2], Zukarnain et al. studied the time evolution of a system comprising of three oscillators *A*, *B* and *C*. The system is described with the Hamiltonian where the oscillator *B* and *C* interact with each other and no interaction between oscillator *A*. Two examples are considered for $\rho_{AB}(0)$, i.e. separable state and entangled state. After the measurement is performed in oscillator *C* whose result is communicated classically to *A*, it is shown for both cases that the effects of the measurements to the separable and entangled states are identical. The measurement effects the density matrix ρ_A of the oscillator *A*. In many cases, ρ_A becomes a pure state and in some cases it becomes a mixed state.

Another comprehensive work done by Ref. [9]. Here, the author investigated the bipartite system which performed the measurement on first subsystem and calculate the entropy of the second subsystem. The entropy is calculated for the three cases. The first case which refer as without measurement, calculates the entropy before the measurement, $S\left(\rho_2\right)$.The second case known as the non-selective measurement, calculates the entropy after the measurement without knowing the outcome, $S\left(\tilde{\rho}_2\right)$. The last case which refer as selective measurement calculates the entropy after the measurement with informed outcome, $S\left(\rho'_2\left(N\right)\right)$ with *N* as the different outcome.

Various cases of state have been considered. These include pure states, separable states, entangled states and entangled states after the symplectic transformation [4]. The results shown in Table 1.

Table 1. The results of bipartite system with non-selective measurement on first subsystem and calculated the entropy of the second subsystem.

States	Non-selective
Pure states	$S(\rho_2) = S(\tilde{\rho}_2) = 0$
Separable states	$S(\tilde{\rho}_2) = S(\rho_2)$
Entangled states	$S(\tilde{\rho}_2) = S(\rho_2)$
Entangled states after symplectic transformations	$S(\tilde{\rho}_2) = S(\rho_2)$

4. Example of Tripartite System

In this paper, we considered an example of a tripartite quantum system based on an example discussed in Ref. [3]. We assumed the example is pure and the factorizable state at $t = 0$ is given by

$$\left|\psi\right\rangle = \frac{1}{2}\left(\left|0\right\rangle_A + \left|1\right\rangle_A\right) \otimes \left|2\right\rangle_B \otimes \left(\left|1\right\rangle_C + \left|2\right\rangle_C\right) \tag{4}$$

The Hamiltonian of the example is

$$
\begin{aligned}
H = {} & \omega_A \left(a_A^\dagger a_A \otimes \mathbf{1} \otimes \mathbf{1} \right) + \omega_B \left(\mathbf{1} \otimes a_B^\dagger a_B \otimes \mathbf{1} \right) + \omega_C \left(\mathbf{1} \otimes \mathbf{1} \otimes a_C^\dagger a_C \right) \\
& + \lambda_{AB} \left[a_A \otimes \left(a_B^\dagger \right)^2 \otimes \mathbf{1} \right] + \lambda_{AB}^* \left[a_A^\dagger \otimes a_B^2 \otimes \mathbf{1} \right] \\
& + \lambda_{BC} \left[\mathbf{1} \otimes a_B^\dagger \otimes a_C^2 \right] + \lambda_{BC}^* \left[\mathbf{1} \otimes a_B \otimes \left(a_C^\dagger \right)^2 \right]
\end{aligned} \tag{5}
$$

where ω_A, ω_B and ω_C are frequencies of the three oscillators. λ_{AB} and λ_{BC} are coupling constants for oscillators A, B and oscillators B, C, respectively. In this case we consider $\omega_A = 2\omega_B = 4\omega_c = 1$ and $\lambda_{AB} = \lambda_{BC} = 1$. With this situation, we expect strong coupling between all three oscillators.

With the initial state at $t = 0$, we assume that the density matrix is $\rho_{ABC}(0)$. Then at time t the density matrix evolves according to the following

$$
\rho_{ABC}(t) = e^{iHt} \rho_{ABC}(0) e^{-iHt} \tag{6}
$$

In this research, we investigated the effects of quantum measurement on a chain of three oscillators A, B and C with nearest neighbour coupling. We performed a photon number N measurements on A and the measurement takes place at $\omega_A t = 3$. Projection operators $\prod_N$ which is performed on the oscillator A is

$$
\prod_N = \prod_N^A \otimes \mathbf{1}_B \otimes \mathbf{1}_C \equiv \left| N \right\rangle \left\langle N \right| \otimes \mathbf{1}_B \otimes \mathbf{1}_C \tag{7}
$$

where N is a photon number or outcomes and $\mathbf{1}_i$ is an identity matrix for the subsystems. We considered the strong coupling between the oscillators A, B and oscillators B, C. There are no direct coupling between oscillator A, C. We calculated the entropy $\left(S(\rho) \right)$, quantum mutual information (I_{ij}) and conditional entropy $\left(E(i \mid j) \right)$ before and after the measurements. We also calculate the partial trace by

$$
\rho_{ij} = \mathrm{Tr}_k \left(\rho_{ijk} \right);\ \rho_i = \mathrm{Tr}_{jk} \left(\rho_{ijk} \right);\ i, j, k = A, B, C \tag{8}
$$

Entropy of the density matrix is calculated by von Neumann entropy as

$$
S(\rho) = -\mathrm{Tr}\left[\rho \log \rho \right] \tag{9}
$$

In this Eq. (9), we used logarithm with base two, so all the entropic quantities are in bits. If we assume λ_i as the eigenvalue of ρ then we can re-expressed the von Neumann entropy as

$$
S(\rho) = -\sum_i \lambda_i \log\left(\lambda_i \right) \tag{10}
$$

In our calculation, we used Eq. (10) to calculate the entropy with log to the base of 2.

Quantum mutual information, I_{ij} is used to measure classical and quantum correlation between two subsystems

$$
I_{ij} = S\left(\rho_i \right) + S\left(\rho_j \right) - S\left(\rho_{ij} \right) \tag{11}
$$

If $I_{ij} \geq 0$, it indicates existing classical and quantum correlation between two subsystems. Entanglement is measured by using an entanglement witness called conditional entropy, $E(i \mid j)$

$$
E(i \mid j) = S\left(\rho_{ij} \right) - S\left(\rho_j \right) \tag{12}
$$

Negative values of $E(i\,|\,j)$ shows that there exist entanglements between two subsystems whilst positive values give an inconclusive witness result.

5. Numerical Results

In this example, we performed a photon number measurement on A and the measurement takes place at $\omega_A t = 3$. We investigated two cases, without measurement and non-selective measurement. Without measurement means that we do not perform any measurement on density matrix after the evolution. Then we investigated the correlation and entanglement of the tripartite quantum systems for all cases by calculating the entropy, quantum mutual information and conditional entropy using similar approach as explained in Ref. [3].

5.1. Entropy

Referring to Fig. 1, it is shown that

$$S\left(\tilde{\rho}_i\left(t\right)\right) \geq S\left(\rho_i\left(t\right)\right);\ i = A, B, C \tag{13}$$

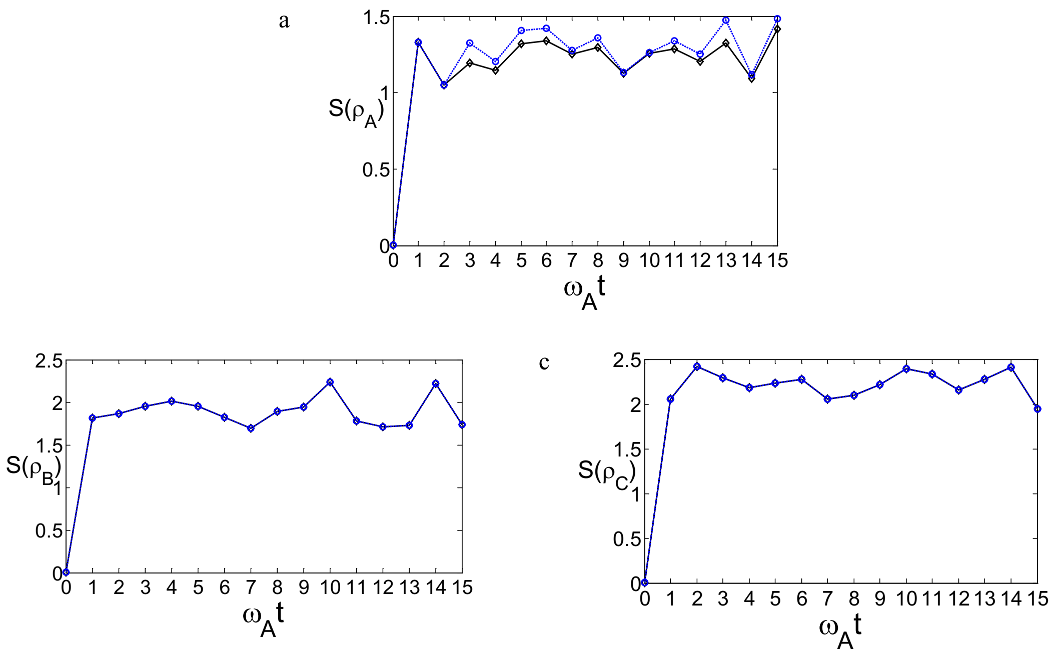

Fig. 1. The entropy of oscillator *A, B* and *C* as a function of time $\omega_A t$. The measurement takes place at $\omega_A t = 3$. (a) we show $S\left(\rho_A\left(t\right)\right)$ (solid line and diamond) and $S\left(\tilde{\rho}_A\left(t\right)\right)$ (dashed line and circle); (b) we show $S\left(\rho_B\left(t\right)\right)$ (solid line and diamond) and $S\left(\tilde{\rho}_B\left(t\right)\right)$ (dashed line and circle); (c) we show $S\left(\rho_C\left(t\right)\right)$ (solid line and diamond) and $S\left(\tilde{\rho}_C\left(t\right)\right)$ (dashed line and circle).

the entropy of density matrix $i = A, B, C$ of non-selective measurement is greater than or equal to the entropy of density matrix $i = A, B, C$ without measurement. The entropy of oscillator B and oscillator C for non-selective measurement are equal with entropy of oscillator B and oscillator C without measurement. This equality can happen if and only if [8]

$$\tilde{\rho}_B\left(t\right) = \rho_B\left(t\right) \quad ; \quad \tilde{\rho}_c\left(t\right) = \rho_c\left(t\right) \tag{14}$$

the density matrix after non-selective measurement equals with density matrix before the measurement.

5.2. Correlation

Looking at Fig. 2, it showed that

$$\tilde{I}_{ij}(t) \geq 0;\ i, j = A.B, C \tag{15}$$

It indicates that there exists classical and quantum correlation among all the oscillators A, B and C. It also showed that

$$\tilde{I}_{ij}(t) \leq I_{ij}(t);\ i, j = A, B, C \tag{16}$$

This means that for non-selective measurement it decreases the correlation.

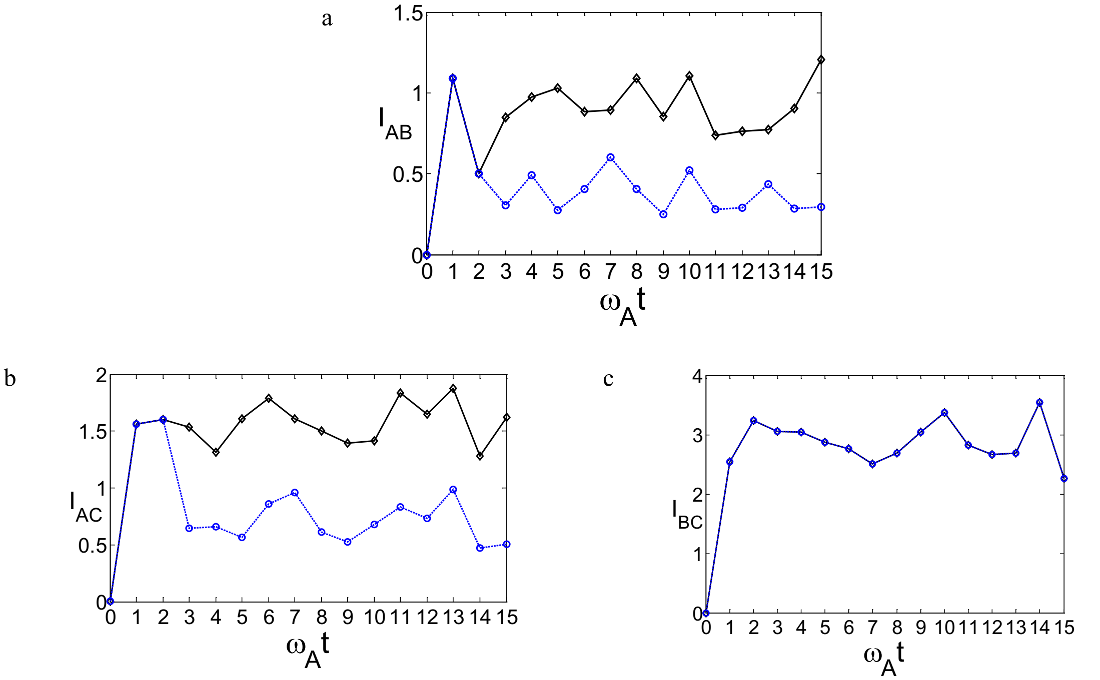

Fig. 2. The quantum mutual information I_{AB}, I_{AC} and I_{BC} as a function of time $\omega_A t$. The measurement takes place at $\omega_A t = 3$. (a) we show $I_{AB}(t)$ (solid line and diamond), $\tilde{I}_{AB}(t)$ (dashed line and circle); (b) we show $I_{AC}(t)$ (solid line and diamond), $\tilde{I}_{AC}(t)$ (dashed line and circle); (c) we show $I_{BC}(t)$ (solid line and diamond), $\tilde{I}_{BC}(t)$ (dashed line and circle).

5.3. Entanglement

Referring to Fig. 3 and Fig. 4 it shows that

$$\tilde{E}(A \mid B)(t) > 0; \tilde{E}(B \mid A)(t) > 0 \tag{17}$$

$$\tilde{E}(A \mid C)(t) > 0; \tilde{E}(C \mid A)(t) > 0 \tag{18}$$

The quantities $E(A|B)$, $E(B|A)$ and $E(A|C)$, $E(C|A)$ are positive for non-selective measurement. With this result, we cannot conclude whether the oscillators A, B and oscillators A, C are entangled or not.

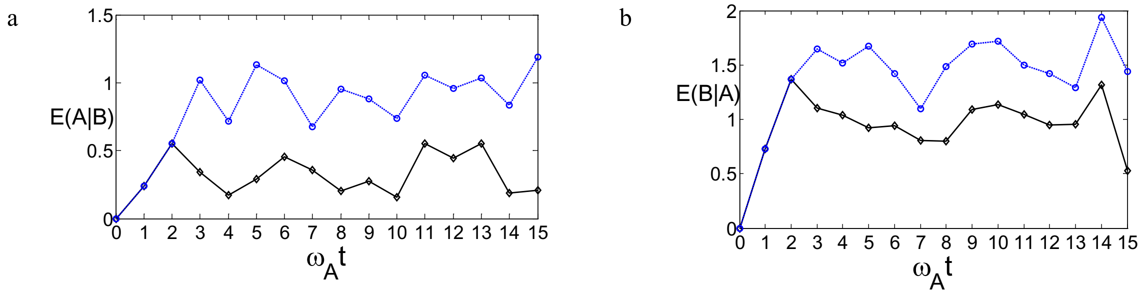

Fig. 3. The conditional entropy $E(A|B)$ and $E(B|A)$ as a function of time $\omega_A t$ .The measurement takes place at $\omega_A t = 3$. (a) we show $E(A|B)$(t) (solid line and diamond), $\tilde{E}(A|B)(t)$ (dashed lineand circle); (b) we show $E(B|A)(t)$ (solid line), $\tilde{E}(B|A)(t)$ (dashed line and circle)

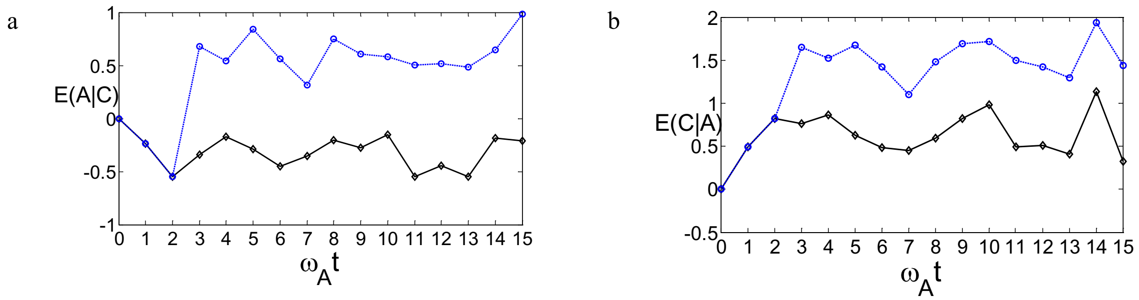

Fig. 4. The conditional entropy $E(A|C)$ and $E(C|A)$ as a function of $\omega_A t$ .The measurement takes place at $\omega_A t = 3$. (a) we show $E(A|C)(t)$ (solid line and diamond), $\tilde{E}(A|C)(t)$ (dashed line and circle); (b) we show $E(C|A)(t)$ (solid line and diamond), $\tilde{E}(C|A)(t)$ (dashed line and circle)

Next, by looking at Fig. 5, it showed that the quantities $E(B|C)$ and $E(C|B)$ are negative for non-selective measurement. It shows that oscillators *B, C* are entangled. With the assumption that the more negative the conditional entropy, the more the entanglement.

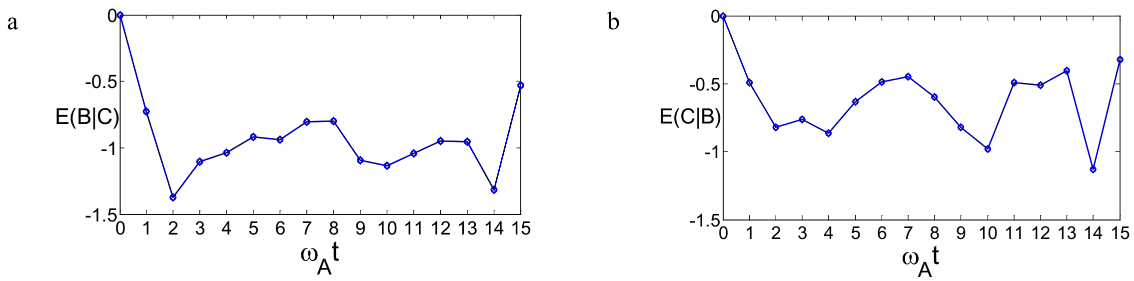

Fig. 5. The conditional entropy $E(B|C)$ and $E(C|B)$ as a function of $\omega_A t$ .The measurement takes place at $\omega_A t = 3$. (a) we show $E(B|C)(t)$ (solid line and diamond), $\tilde{E}(B|C)(t)$ (dashed line and circle); (b) we show $E(C|B)(t)$ (solid line and diamond), $\tilde{E}(C|B)(t)$ (dashed line and circle)

6. Discussion

Investigating quantum measurement in various components of a multipartite quantum system has been an interesting research, especially when it changes the state of the system. In the numerical calculation, it is shown that the entropy of oscillators *A, B* and *C* after the non-selective measurement is non-negative and greater than the entropy without measurement. This is an expected result because from the theorem, the entropy after the non-selective measurement is at least as great as the original entropy. This shows that our results are in line with the theorem.

We also calculated the quantum mutual information I_{ij} for the case of after measurements. In many cases, the results prove that there exist classical and quantum correlation among all of the oscillators *A*, *B* and *C*. Non-selective measurement shows that the measurement decrease the correlations of oscillators *A, B* and oscillators *A*, *C* (refer to Fig. 2). Meanwhile, the quantum mutual information for oscillators *B*, *C* is equal before and after the measurement. This indicates that the measurement does not effect the correlation of oscillators *B*, *C*. We also note that the quantum mutual information for oscillators *A, C* is still greater than *A*, *B*, which is the same as before the measurement.

We also proved that, based on conditional entropy calculation, after the non-selective measurements, oscillators *A*, *B* and oscillators *A, C* are inconclusive. Meanwhile, oscillators *B*, *C* maintain the entanglement.

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